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Washington Observations	(1874) 284 - 287
	(1875) 356 - 361
	(1876) 389 - 393
	(1877) 227 - 229
	(1878) 92 - 93
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	(1881) 110 - 111
	(1882) 108
	(1883) 136 - 138
	(1884) 204 - 206

(1885) 177

(1886) 129 - 130

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(1888) 21

(1889) 95 - 96

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```
// Expansion of the Solar disturbing function in terms of the mutual
// inclination
//
//
// a = sin(I/2)
//
// e = eccentricity of satellite
// f = eccentricity of the Sun
//
// s = true anomaly of satellite
// t = true anomaly of the Sun
// u = apse angle of satellite
// v = apse angle of the Sun
// w = mean anomaly of satellite
// x = mean anomaly of the Sun
//
// define maximum order for expansion
//
WEIGHT(a)=0
WEIGHT(e)=1
WEIGHT(f)=1
MAXORDER=3
//
// First define cosS by spherical trigonometry
//
A=cos[s+u]cos[t+v]+(1-2a.2)sin[s+u]sin[t+v]
//
// The equation of the centre for the satellite
//
B=(2e-(1/4)e.3)sin[w] +((5/4)e.2-(11/24)e.4)sin[2w] +(13/12)e.3sin[3w]
B:=B+ (103/96)e.4sin[4w]
//
// The equation of the centre for the Sun
//
C=(2f-(1/4)f.3)sin[x] +((5/4)f.2-(11/24)f.4)sin[2x] +(13/12)f.3sin[3x]
C:=C+ (103/96)f.4sin[4x]
//
// The radius vector for the satellite (actually r/a)
//
D=1+(1/2)e.2 +(-e+(3/8)e.3)cos[w] +(-(1/2)e.2+(1/3)e.3)cos[2w]
D:=D- (3/8)e.3cos[3w] - (1/3)e.4cos[4w]
//
// The inverse radius vector of the Sun, in terms of the true anomaly
//
E=(1+f.2+f.4)(1+fcos[t])
```

```
//
// Now we substitute the equation of the centre to eliminate true anomalies
//
E:=HSUB(E,t,x,C,4)
//
A:=HSUB(A,s,w,B,4)
A:=HSUB(A,t,x,C,4)
//
// and multiply everything together
//
G:=DDEEE(-1+3AA)/2
PAGE
TEXT: Solar disturbing function:
PRINT[G]
STOP
END
```

APPENDIX B. FORMULAE FROM SPHERICAL TRIGONOMETRY

In several sections of this thesis, notably chapters 2 and 4, we seek differential relationships between parts of a spherical triangle. In general, we start with a set of formulae such as

$$\cos\vartheta = \alpha$$

$$\sin\vartheta\cos\phi = \beta$$

$$\sin\vartheta\sin\phi = \gamma$$

where ϑ, ϕ are the dependent variables and α, β, γ are functions of the independent variables, say w, x, y et cetera. We seek derivatives such as $\partial\vartheta/\partial w$, $\partial\phi/\partial w$. We may immediately write

$$-\sin\vartheta\, d\vartheta = d\alpha$$

and

$$\cos\vartheta\cos\phi\, d\vartheta - \sin\vartheta\sin\phi\, d\phi = d\beta$$

$$\cos\vartheta\sin\phi\, d\vartheta + \sin\vartheta\cos\phi\, d\phi = d\gamma$$

whence

$$\cos\vartheta\, d\vartheta = \cos\phi\, d\beta + \sin\phi\, d\gamma$$

$$\sin\vartheta\, d\phi = -\sin\phi\, d\beta + \cos\phi\, d\gamma.$$

Thus

$$\sin\vartheta \frac{d\vartheta}{dw} = -\frac{d\alpha}{dw}$$

$$\cos\vartheta \frac{d\vartheta}{dw} = \cos\phi \frac{d\beta}{dw} + \sin\phi \frac{d\gamma}{dw}$$

$$\sin\vartheta \frac{d\phi}{dw} = -\sin\phi \frac{d\beta}{dw} + \cos\phi \frac{d\gamma}{dw}.$$

As an example, we shall calculate the partial derivatives of Chapter 2, equation [21]. We may write the following formulae (cf. Explanatory Supplement to the Astronomical Ephemeris (1961) page 472).

(a) $\sin i \sin(\Omega - \Omega_s) = \sin\eta \sin(\Theta - \Omega_s)$

(b) $\sin i \cos(\Omega - \Omega_s) = \sin i_s \cos\eta + \cos i_s \sin\eta \cos(\Theta - \Omega_s)$

(c) $\cos i = \cos i_s \cos\eta - \sin i_s \sin\eta \cos(\Theta - \Omega_s)$

(d) $\sin i \sin\vartheta = \sin i_s \sin(\Theta - \Omega_s)$

(e) $\sin i \sin\vartheta = \cos i_s \sin\eta + \sin i_s \cos\eta \cos(\Theta - \Omega_s)$

From (c) we derive

$$\begin{aligned} -\sin i \, di/d\eta &= -\cos i_s \sin\eta - \sin i_s \sin\eta\cos(\Theta - \Omega_s) \\ &= -\sin i \cos\vartheta. \end{aligned}$$

Thus $di/d\eta = \cos\vartheta$.

Also,

$$- \sin i \, di/d\Theta = \sin i_s \sin \eta \sin(\Theta - \Omega_s)$$
$$= \sin \eta \sin i \sin \vartheta.$$

Thus $\quad di/d\Theta = - \sin \vartheta \sin \eta.$

We introduce three further formulae.

(f) $\quad \sin \vartheta \cos \eta = \sin(\Theta - \Omega_s) \cos(\Omega - \Omega_s) - \cos(\Theta - \Omega_s) \sin(\Omega - \Omega_s) \cos i_s$

(g) $\quad \sin \vartheta \sin \eta = \sin i_s \sin(\Omega - \Omega_s)$

(h) $\quad \cos \vartheta = \cos(\Theta - \Omega_s) \cos(\Omega - \Omega_s) + \sin(\Theta - \Omega_s) \sin(\Omega - \Omega_s) \cos i_s$

and from (a) and (b) we obtain

$$\sin i \frac{d\Omega}{d\eta} = - \sin(\Omega - \Omega_s) \{-\sin i_s \sin \eta + \cos i_s \cos \eta \cos(\Theta - \Omega_s)\}$$
$$+ \cos(\Omega - \Omega_s) \{\cos \eta \sin(\Theta - \Omega_s)\}$$

$$= \sin \vartheta$$

and

$$\sin i \frac{d\Omega}{d\Theta} = - \sin(\Omega - \Omega_s) \{- \cos i_s \sin \eta \cos(\Theta - \Omega_s)\}$$

$$+ \cos(\Omega - \Omega_s) \{\sin\eta \cos(\Theta - \Omega_s)\}$$

$$= \sin\eta \sin\vartheta.$$

From (d) and (e) we have

$$\sin i \frac{d\vartheta}{d\eta} = - \sin\vartheta \{\cos i_s \cos\eta - \sin i_s \sin\eta \sin(\Theta - \Omega_s)\}$$

$$= - \sin\vartheta \cos i$$

and

$$\sin i \frac{d\vartheta}{d\Theta} = - \sin\vartheta \{- \sin i_s \cos\eta \sin(\Theta - \Omega_s)\}$$

$$+ \sin\vartheta \{\sin i_s \cos(\Theta - \Omega_s)\}$$

$$= \sin i_s \sin(\Omega - \Omega_s).$$

Thus we have the derivatives quoted in Chapter 2.

APPENDIX C. INDIRECT TERMS IN SOLAR AND SATELLITE PERTURBATIONS

In the Solar disturbing function of Chapter 2, and the contributions to the force model of Chapter 5 by the Sun and by mutual satellite perturbations, there appear terms known as indirect terms in addition to the direct inverse-square term. The acceleration of a satellite due to the action of the Sun is thus

$$\underline{a}_{is} = G M_s \left\{ \frac{\underline{r}_s - \underline{r}_i}{r_{is}^3} - \frac{\underline{r}_s}{r_s^3} \right\}$$

where G = the gravitational constant

M_s = mass of the Sun

$\underline{r}_s$ = Saturnicentric position vector of the Sun

$\underline{r}_i$ = Saturnicentric position vector of the i^{th} satellite

$r_{is} = | \underline{r}_s - \underline{r}_i |$

$r_s = | \underline{r}_s |$.

The first term is the direct gravitational attraction while the second term may be interpreted as the acceleration of the Sun relative to Saturn. It arises because the coordinate system is referred to the centre of Saturn (i.e. the centre of mass of Saturn, not of the entire system) and is thus not inertial.

Consider the equations of motion of the system consisting of Saturn, its satellites and the Sun, referred to an inertial frame. Then

(a) $$M_o\ddot{\underline{R}}_o = -GM_oM_s\left(\frac{\underline{R}_o - \underline{R}_s}{R_{os}^3}\right) - GM_om_i\left(\frac{\underline{R}_o - \underline{R}_i}{R_{oi}^3}\right)$$

where M_o = mass of Saturn

M_s = mass of the Sun

m_i = mass of the i^{th} satellite

$\underline{R}_o$ = position vector of Saturn

$\underline{R}_s$ = position vector of the Sun

$\underline{R}_i$ = position vector of the i^{th} satellite

and $$R_{os} = |\underline{R}_o - \underline{R}_s|$$

$$R_{oi} = |\underline{R}_o - \underline{R}_i|.$$

For the i^{th} satellite we have

(b) $$M_i\ddot{\underline{R}}_i = -GM_oM_i\left(\frac{\underline{R}_i - \underline{R}_o}{R_{io}^3}\right) - GM_sm_i\left(\frac{\underline{R}_s - \underline{R}_i}{R_{is}^3}\right) - \sum_{j}^{j\neq i} Gm_iM_j\left(\frac{\underline{R}_i - \underline{R}_j}{R_{ij}^3}\right)$$

where $\underline{R}_j$ = position vector of the j^{th} satellite.

We seek the acceleration of the i^{th} satellite relative to Saturn and thus we want

$$\ddot{\underline{r}}_i = \ddot{\underline{R}}_i - \ddot{\underline{R}}_o$$

where we use upper-case letters to denote position vectors in the inertial frame and lower-case letters to denote those in the Saturnicentric frame. Thus

$$\ddot{\underline{R}}_i = -G(M_o + m_i)\,\frac{\underline{R}_i - \underline{R}_o}{R_{io}^3}$$

$$- G\,M_s\left(\frac{\underline{R}_i - \underline{R}_s}{R_{is}^3} + \frac{\underline{R}_s - \underline{R}_o}{R_{os}^3}\right)$$

$$-\sum_{j}^{j\neq i} G\,M_j\left(\frac{\underline{R}_i - \underline{R}_j}{R_{ij}^3} + \frac{\underline{R}_j - \underline{R}_o}{R_{oj}^3}\right)$$

$$= -\;G(M_o + m_i)\,\frac{\underline{r}_i}{r_{io}^3}$$

$$-\;G\,M_s\left(\frac{\underline{r}_i - \underline{r}_s}{r_{is}^3} + \frac{\underline{r}_s}{r_{os}^3}\right)$$

$$-\sum_{j}^{j\neq i} G\,M_j\left(\frac{\underline{r}_i - \underline{r}_j}{r_{ij}^3} + \frac{\underline{r}_j}{r_{oj}^3}\right)$$

The origin of the indirect terms may be clearly seen from this derivation. In chapter 2, we also require the disturbing function for Solar perturbations. This is the function R_s such that

$$\text{grad } R_s = G\,M_s\left(\frac{\underline{r}_s - \underline{r}_i}{r_{is}^3} - \frac{\underline{r}_s}{r_s^3}\right).$$

The direct term is evidently GM/r_{is}. The indirect term in the acceleration does not contain the coordinates of the perturbed satellite and thus its contribution to the disturbing function may be written as

$$- G M_s \frac{\{x x_s + y y_s + z z_s\}}{r_s^3}$$

$$= \quad - G M_s \frac{\underline{r}.\underline{r}_s.}{r_s^3}$$

The Solar disturbing function is thus

$$R_s = G M_s \left(\frac{1}{r_{is}} - \frac{\underline{r}.\underline{r}_s.}{r_s^3} \right)$$

Recalling that $\underline{r}.\underline{r}_s = r\ r_s \cos X$ where X is the angle subtended at the centre of Saturn by the position vectors $\underline{r}$ and $\underline{r}_s$, we may write this as

$$R_s = G M_s \left(\frac{1}{r_{is}} - \frac{r \cos X}{r_s^2} \right)$$

(cf. Brouwer and Clemence (1961) page 308, equation (1a))

APPENDIX D. EXPRESSIONS FOR OBLATENESS PERTURBATIONS

In chapter 5 we require the components of the force upon each satellite due to the oblateness of Saturn. We begin by considering the disturbing function for oblateness perturbations

(a) $$R_e = - GM/r \quad (a_e/r)^n J_n P_n(w)$$

where a_e = equatorial radius of Saturn

r = distance from the centre of Saturn

J_n = n^{th} harmonic coefficient of the potential field

w = z/r = latitude above the equatorial plane of Saturn

and $P_n(w)$ is a Legendre polynomial of degree n.

The acceleration upon a satellite at this point due to the oblateness of Saturn is

(b) $$\underline{a}_e = \nabla R_e .$$

Since Saturn is symmetrical about its equatorial plane, we may neglect all odd harmonics. The first term is thus the J_2 term. Consider the n^{th} harmonic term

(c) $$R_n = - (GM/r) J_n (a_e/r)^n P_n(w).$$

The components of the acceleration of the satellite due to this term are then

$$\ddot{x}_n = \partial R_n/\partial x$$
$$\ddot{y}_n = \partial R_n/\partial y$$
$$\ddot{z}_n = \partial R_n/\partial z.$$

Now we may write

$$\partial R_n/\partial x = + G\ M\ J_n\ (a_e/r)^n\ (x/r^3)\ \{(n+1)P_n(w) + w.P'_n(w)\}$$

$$= G\ M\ J_n\ (a_e/r)^n\ (x/r^3)\ P'_{n+1}(w).$$

The expression for $\partial R_n/y$ is exactly similar, writing y in place of x, while the expression for $\partial R_n/\partial z$ has an extra term

$$- G\ M\ J_n\ (a_e/r)^n\ (1/r^2)\ P'_n(w)$$

because w depends upon z directly, as well as indirectly via r. Thus we may write

$$\underline{a}_n = G\ M\ J_n(a_e/r)^n\ (1/r^2)\ \{P'_{n+1}(w)\ \ \underline{r} - P'_n(w)\ \ \underline{k}\}$$

where $\underline{r}$ is a unit vector in the radial direction

$\underline{k}$ is the unit vector in the z direction.

This is the form of the oblateness accelerations used by Sinclair and Taylor (1985).

APPENDIX E. FINAL SETS OF PARAMETERS FROM NUMERICAL INTEGRATION TRIALS

Epoch = Julian Ephemeris Date 2418800.5

	Trial 1	Trial 2	Trial 3
Titan			
	-0.0079438545	-0.0079440917	-0.0079429489
$\underline{r}$	0.0002251206	0.0002253203	0.0002302996
	-0.0000197461	-0.0000195983	-0.0000191867
	-0.0001257187	-0.0001257718	-0.0001268372
$\dot{\underline{r}}$	-0.0033045519	-0.0033044504	-0.0033048230
	0.0000183595	0.0000185320	0.0000186238
Hyperion			
	0.0058500907	0.0058503911	0.0058519167
$\underline{r}$	-0.0093650299	-0.0093636307	-0.0093600560
	0.0000713479	0.0000720276	0.0000710875
	0.0021938372	0.0021938579	0.0021942391
$\dot{\underline{r}}$	0.0013940284	0.0013944440	0.0013949571
	-0.0000290135	-0.0000288438	-0.0000288451
Iapetus			
	-0.0226958048	-0.0226958837	-0.0226951800
$\underline{r}$	0.0015958220	0.0015959033	0.0015987369
	-0.0056343356	-0.0056344345	-0.0056342563
	-0.0001092249	-0.0001092299	-0.0001093554
$\dot{\underline{r}}$	-0.0019102618	-0.0019102521	-0.0019102873
	0.0000870654	0.0000870693	0.0000870921
Masses and form-factors			
J_2	0.01675414 (*)	0.01675414 (*)	0.017788155
J_4	-0.00100000 (*)	-0.00100000 (*)	-0.00100000 (*)
μ_{Titan}	0.00023678 (*)	0.00023666	0.00023665
$\mu_{Iapetus}$	0.000003308 (*)	0.000003308 (*)	0.000003308 (*)
μ_{Saturn}	0.00028588 (*)	0.00028588 (*)	0.00028588 (*)

(*) not determined by fitting to observations : fixed value adopted.